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Free Vibration of Unsymmetrically Laminated Beams Having Uncertain Ply Orientations

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Three models were developed to predict randomness in the free vibration response of unsymmetrically laminated beams: exact Monte Carlo simulation, sensitivity-based Monte Carlo simulation, and probabilistic finite element analysis (FEA). It is assumed that randomness in the response is only caused by uncertainties in the ply orientations. The ply orientations may become random or uncertain during the manufacturing process. A new 16-degree-of-freedom beam element, based on the first-order shear deformation beam theory, is adapted for use in studying the probabilistic nature of the natural frequencies. With the use of variational principles, the element stiffness matrix and mass matrix are obtained through analytical integration. With the use of a random sequence, a large data set following normal distribution is generated, containing possible random ply orientations. The sensitivity derivatives are numerically calculated through an exact analytical formulation. The eigenvalues are expressed in terms of deterministic and probabilistic quantities, which allows the determination of how sensitive they are to variations in ply angles. The predicted mean value and coefficient of variation of the natural frequencies for sensitivity-based Monte Carlo simulation and probabilistic FEA are in good agreement with the exact Monte Carlo simulation. Results show that variations of ± 5 deg in ply angles have little effect on the lower mode natural frequencies of unsymmetrically and symmetrically laminated beams.

Introduction

IN recent years, there has been an increasing demand for fiber-reinforced laminated composite materials in aircraft structures. The main reasons are that the composites possess the following beneficial characteristics: They are lightweight, cost effective, and able to handle different strengths in different directions.[1] However, these materials offer quite a few challenges to structural engineers. Because of the inherent complexity of fiber-reinforced laminated composites, it can be challenging to manufacture composite structures according to their exact design specifications, resulting in unwanted material and geometric uncertainties.

The design and analysis using conventional materials is easier than those using composites because for conventional materials both material and geometric properties have either little or well-known variation from their nominal value. On the other hand, the same cannot be said for the design of structures using laminated composite materials. Thus, the understanding of uncertainties in laminated structures is very important for an accurate design and analysis of aerospace and other structures. Elishakoff[2] has suggested three different approaches to study uncertainties: 1) probabilistic methods, 2) fuzzy set or possibility-based methods, and 3) antioptimization.

The nonprobabilistic methods such as fuzzy set theory and antioptimization are used when the data regarding the uncertain parameter are not available or little is known about the probability density function (PDF).[3,4] However, these uncertainties are ignored in this research because nonprobabilistic methods are beyond the scope of the present work.

In this research, the noncognitive sources of uncertainty are of great interest and are dealt with using probabilistic methods or nonprobabilistic methods. The noncognitive sources of uncertainty, that is, material and geometric variations, are in general quantified, and information about the uncertainty of these parameters may be available. When sufficient data are available to predict the PDF, then a probabilistic method can be used. Thus, throughout this work, uncertainties due to noncognitive sources are studied using a probabilistic approach.

The randomness of noncognitive sources leads to variations in the stiffness and mass coefficients of the laminates. These uncertainties may involve geometric quantities, that is, ply orientations and dimensions; material properties, that is, elastic modulus, shear modulus, Poisson's ratio, and material density; and external properties, that is, thermal and loading effects. However, in this research only those uncertainties involving geometric properties (ply orientations) are considered.

What has attracted many researchers to study structures with laminated composites is the complexity of these materials. Because it is costly to analyze a composite structure as a three-dimensional solid, analysis of many composite structures can be performed using laminated one- and two-dimensional theories, such as beam theory. A review of various available theories for analyzing laminated beams is given by Kapania and Raciti.[5] Earlier, a 12-degree-of-freedom (DOF) element was developed and formulated for deterministic symmetric laminated beams to study their static and dynamic behaviors.[6] Later a 20-DOF element (Kapania–Raciti element) was developed to study static, free vibration, buckling, and nonlinear vibrational analysis of unsymmetrically laminated beams.[7,8] In these works, the effects of uncertainties were not considered.

The probabilistic analysis can be performed using either an analytical or a computational approach. An analytical approach would be most accurate, although cumbersome and impractical except for very simple systems. However, with the availability of high-speed computers, the finite element method has become a standard for engineers to analyze structures with complex geometry, including various sources of nonlinearities. However, the deterministic finite element method fails to take into account uncertainty in different parameters of the structure, and, thus, cannot be used for reliability analysis.

Various methods exist to analyze an uncertain structure by integrating probabilistic aspects into the finite element modeling. In particular, there has been a growing interest in applying these methods to better understand laminated composite structures by integrating the stochastic nature of the structure in the finite element analysis.[9] When the probabilistic nature of material properties, geometry, and/or loads is integrated in the finite element method, such concept is called probabilistic finite element method.

The probabilistic finite element analysis (PFEA) can be classified into two categories: perturbation techniques and simulation

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methods. Perturbation techniques are based on series expansion, that is, Taylor series, to formulate a linear or quadratic relationship between the randomness of the material, geometry, or load and the randomness of the response (see Refs. 10 and 11). Therefore, an FEA using Monte Carlo simulation is developed to take into account the stochastic nature of the ply orientations.

Work done by Vinckenroy et al.[12] presents a new technique to analyze these structures by combining the stochastic analysis and the finite element method in structural design. Mei et al.[13] used a wavelet-based stochastic analysis to analyze isotropic beam structures.

Oh and Librescu[14] studied the free vibration and reliability of cantilever composite beams featuring structural uncertainties. They used a stochastic Rayleigh–Ritz formulation. However, to the best of the authors' knowledge, no work has been done regarding the effect of uncertainties incurring in the ply orientations on the natural frequencies of unsymmetrically laminated beams using PFEA.

The primary goal of the present work is to predict the dynamic behavior of unsymmetrically laminated beams with uncertainties. To study the stochastic nature of the dynamic response of such beams, a new 16-DOF element is developed using first-order shear deformation beam theory to account for uncertainties. A detailed overview of this element is presented. When this element is used, the free vibration analysis is performed for only those uncertainties associated with layerwise ply orientations.

16-DOF Laminated Beam Element

Overview

The present 16-DOF laminated beam element takes into account the existence of various coupling effects, which play a major roll in laminate composite materials. This element is valid for the analysis of both symmetric and unsymmetrically laminated beams. The motivation to develop a new beam element was to have a formulation consistent with the first-order shear deformation beam theory (FSDT) that is able to analyze unsymmetrically laminated beams and that would account for most of the uncertainties involved in a laminated beam when modeled using FSDT. In the present work, the reference system of the coordinates is such that the x axis lies along the length of the beam and the z axis is placed at the midsurface measuring the transverse displacements. The present work assumes that the x–z plane divides the beam in two identical parts: In other words, material, geometry, and loading are symmetric about the x–z plane. When twisting is considered and in-plane shear is ignored, the displacement field for the FSDT in the defined reference system can be expressed as follows:

$$U(x, y, z) = u(x) + z\phi(x) \tag{1a}$$

$$V(x, y, z) = -z\tau(x) \tag{1b}$$

$$W(x, y, z) = w(x) + y\tau(x) \tag{1c}$$

Therefore, the finite element formulation considers eight DOF at each node: axial displacement u, transverse deflection w, rotation of the transverse normals ϕ, twist angle τ, and their derivatives with respect to x. These nodal displacements measured at the midsurface are

$$\boldsymbol{q}_{t_i}^T = \{u_1, u_1', w_1, w_1', \tau_1, \tau_1', \phi_1, \phi_1', u_2, u_2', w_2, w_2', \tau_2, \tau_2', \phi_2, \phi_2'\} \tag{2}$$

The deflection behavior of the beam element for the first-order theory including transverse twist effect is approximated as

$$u(x) = N_1 u_1 + N_2 u_1' + N_3 u_2 + N_4 u_2'$$

$$w(x) = N_1 w_1 + N_2 w_1' + N_3 w_2 + N_4 w_2'$$

$$\tau(x) = N_1 \tau_1 + N_2 \tau_1' + N_3 \tau_2 + N_4 \tau_2'$$

$$\phi(x) = N_1 \phi_1 + N_2 \phi_1' + N_3 \phi_2 + N_4 \phi_2' \tag{3}$$

where the shape functions $N_i(x)$ are the well-known Hermitian polynomials. This can be expressed as follows

$$\boldsymbol{q}_t(\boldsymbol{x}) = \boldsymbol{N}\boldsymbol{q}_{t_i} \tag{4}$$

where matrix $\boldsymbol{N}$ is known as the shape function matrix.

Strain–Displacement Relationship

The nonzero strains in the preceding formulation are related to the displacements as follows:

$$e_{xx} = \underbrace{\left(\frac{\partial u}{\partial x}\right)}_{\varepsilon_{xx}} + z \underbrace{\left(\frac{\partial \phi}{\partial x}\right)}_{\kappa_{xx}} \tag{5}$$

$$2e_{xz} = \underbrace{\left(\frac{\partial w}{\partial x} + y\frac{\partial \tau}{\partial x} + \phi\right)}_{\gamma_{xz}} \tag{6}$$

$$2e_{xy} = z \underbrace{\left(-\frac{\partial \tau}{\partial x}\right)}_{\kappa_{xy}} \tag{7}$$

In vector form, the strains are expressed as follows:

$$\boldsymbol{\varepsilon} = \{\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \kappa_{xx}, \kappa_{yy}, \kappa_{xy}, \gamma_{xz}\}^T \tag{8}$$

When Eqs. (2), (3), and (8) are used, the strain–displacement relation can be expressed as

$$\boldsymbol{\varepsilon} = \boldsymbol{B}_{\text{sd}}\boldsymbol{q}_{t_i} \tag{9}$$

where $\boldsymbol{B}_{\text{sd}}$ is the strain–displacement matrix.

Constitutive Material Law

The beam element formulation considers orthotropic laminated composites. Although strain e_{zz} is zero, the present formulation will assume state of plane stress ($\sigma_{zz} = 0$) and condense e_{zz} from the stress–strain relationship. The reduced material coefficients are expressed as Q_{ij}.

Because in laminated composites each ply may have different orientation, the stresses are expressed in terms of an arbitrary angle θ, and the transformed plane stress-reduced elastic coefficients are expressed as $\bar{Q}_{ij}$ (Jones[1]). Thus, the transformed stress–strain relationship takes the following form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \\ 2e_{xz} \end{Bmatrix} \tag{10}$$

A, *B*, and *D* Matrices

The extensional matrix $\boldsymbol{A}$, the extensional–bending coupling matrix $\boldsymbol{B}$, and the bending–stiffness matrix $\boldsymbol{D}$ are defined as[15]

$$A_{ij} = \sum_{k=1}^{N_{\text{lam}}} \bar{Q}_{ij}(z_{k+1} - z_k), \qquad i, j = 1, 2, 5, 6 \tag{11}$$

$$B_{ij} = \sum_{k=1}^{N_{\text{lam}}} \bar{Q}_{ij}\left(\frac{z_{k+1}^2 - z_k^2}{2}\right), \qquad i, j = 1, 2, 6 \tag{12}$$

$$D_{ij} = \sum_{k=1}^{N_{\text{lam}}} \bar{Q}_{ij}\left(\frac{z_{k+1}^3 - z_k^3}{3}\right), \qquad i, j = 1, 2, 6 \tag{13}$$

where N_{lam} is the total number of laminates considered. When symmetrically laminated composites are considered, $\boldsymbol{B}$ is identically zero, and the coupling between bending and stretching vanish. However, for unsymmetrically laminated beams, the coupling cannot be ignored, and it must be included in the analysis. In the presence of uncertainties, laminate composite structures are no longer symmetric, and the analysis of unsymmetrically laminated structures is a more accurate one.

Laminate Constitutive Relations

The basic constitutive relation is

$$\boldsymbol{Q}_s = \boldsymbol{C}\varepsilon \tag{14}$$

where $\boldsymbol{Q}_s = \{N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}, Q_x\}^T$ is the stress and moment resultant vector, $\boldsymbol{C}$ is the bending–stiffness matrix, and ε is the strain vector, defined by Eq. (9).

In the analysis of one-dimensional beam elements, the N_{yy} resultant force and the M_{yy} bending moment are made to vanish:

$$M_{yy} = N_{yy} = 0 \tag{15}$$

On the other hand, the displacement field suggests $e_{yy} = 0$. Loads and strains cannot be prescribed at the same time. Because a one-dimensional analysis is being considered, the load conditions are prescribed, and the in-plane strain ε_{yy} and the bending curvature κ_{yy} are condensed in the constitutive equations.

The bending–stiffness matrix is partitioned and renamed $\boldsymbol{D}_c$ as follows:

$$\boldsymbol{D}_c = \begin{bmatrix} \begin{bmatrix} \boldsymbol{D}_{I,I} & \boldsymbol{D}_{I,II} \\ \boldsymbol{D}_{II,I} & \boldsymbol{D}_{II,II} \end{bmatrix} & [0] \\ [0] & [K_s \boldsymbol{A}_{55}] \end{bmatrix} \tag{16}$$

where

$$\boldsymbol{D}_{I,I} = \begin{bmatrix} A_{11} & B_{11} & B_{16} \\ B_{11} & D_{11} & D_{16} \\ B_{16} & D_{16} & D_{66} \end{bmatrix} \tag{17}$$

$$\boldsymbol{D}_{II,I} = \begin{bmatrix} A_{12} & B_{12} & B_{26} \\ B_{12} & D_{12} & D_{26} \end{bmatrix} = \boldsymbol{D}^T_{I,II} \tag{18}$$

$$\boldsymbol{D}_{II,II} = \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix} \tag{19}$$

The reduced form of the first submatrix in the bending–stiffness matrix is calculated as

$$\boldsymbol{D}_R = \boldsymbol{D}_{I,I} - \boldsymbol{D}_{I,II}\boldsymbol{D}^{-1}_{II,II}\boldsymbol{D}_{II,I} \tag{20}$$

When the preceding expression is used, an equivalent bending–stiffness matrix $\boldsymbol{D}_c$ for an unsymmetrically laminated beam is determined to be

$$\boldsymbol{D}_c = \begin{bmatrix} D_{c_{11}} & D_{c_{12}} & D_{c_{13}} & 0 \\ D_{c_{12}} & D_{c_{22}} & D_{c_{23}} & 0 \\ D_{c_{13}} & D_{c_{23}} & D_{c_{33}} & 0 \\ 0 & 0 & 0 & D_{c_{44}} \end{bmatrix} \tag{21}$$

where the coefficients of $\boldsymbol{D}_c$ are

$$D_{c_{11}} = \big(A_{22}B^2_{12} - 2A_{12}B_{12}B_{22} + A_{11}B^2_{22} + A^2_{12}D_{22} - A_{11}A_{22}D_{22}\big)\big/\Delta$$

$$D_{c_{12}} = \big(-B^2_{12}B_{22} + B_{11}B^2_{22} + A_{22}B_{12}D_{12} - A_{12}B_{22}D_{12} - A_{22}B_{11}D_{22} + A_{12}B_{12}D_{22}\big)\big/\Delta$$

$$D_{c_{13}} = \big(B_{16}B^2_{22} - B_{12}B_{22}B_{26} - A_{22}B_{16}D_{22} + A_{12}B_{26}D_{22} + A_{22}B_{12}D_{26} - A_{12}B_{22}D_{26}\big)\big/\Delta$$

$$D_{c_{22}} = \big(B^2_{22}D_{11} - 2B_{12}B_{22}D_{12} + A_{22}D^2_{12} + B^2_{12}D_{22} - A_{22}D_{11}D_{22}\big)\big/\Delta$$

$$D_{c_{23}} = \big(-B_{22}B_{26}D_{12} + B^2_{22}D_{16} + B_{12}B_{26}D_{22} - A_{22}D_{16}D_{22} - B_{12}B_{22}D_{26} + A_{22}D_{12}D_{26}\big)\big/\Delta$$

$$D_{c_{33}} = \big(B^2_{26}D_{22} - 2B_{22}B_{26}D_{26} + A_{22}D^2_{26} + B^2_{22}D_{66} - A_{22}D_{22}D_{66}\big)\big/\Delta$$

$$D_{c_{44}} = K_s A_{55}, \qquad \Delta = B^2_{22} - A_{22}D_{22}$$

K_s is the transverse shear correction factor. First-order Mindlin theory assumes that the transverse shear strain and, thus, the transverse shear stresses are constant along the thickness direction. For this case, Reissner[16] proposed $K = \frac{5}{6}$ as an approximation to the shear correction factor. However, the actual variation of the transverse shear stresses is not constant, and this can lead to significant discrepancies for unsymmetrically laminated beams.[17] Currently the authors of this paper are expanding their formulation to nonlinear analysis including warping and a more accurate transverse shear distribution. However, here we take the shear correction factor as $\frac{5}{6}$.

Element Stiffness Matrix

When Eqs. (9) and (14) are used, and when it is noted that matrix $\boldsymbol{D}_c$ has been integrated throughout the thickness, the virtual work done by internal forces in a beam element can be written as

$$\delta\mathcal{W}_{\text{int}} = \delta\boldsymbol{q}^T_{t_i} \underbrace{\left[\iint_\Omega \boldsymbol{B}^T_{\text{sd}}\boldsymbol{D}_c\boldsymbol{B}_{\text{sd}}\,\mathrm{d}\Omega\right]}_{K^e} \boldsymbol{q}_{t_i} \tag{22}$$

where $\boldsymbol{K}^e$ is the element stiffness matrix

$$\boldsymbol{K}^e = \int_0^{\ell_e}\int_{-b_0/2}^{b_0/2} \boldsymbol{B}^T_{\text{sd}}\boldsymbol{D}_c\boldsymbol{B}_{\text{sd}}\,\mathrm{d}y\,\mathrm{d}x \tag{23}$$

and b_0 is the width of the beam, ℓ_e is the length of the beam element, $\boldsymbol{D}_c$ is the equivalent bending–stiffness matrix.

Element Mass Matrix

The virtual kinetic energy can be expressed as

$$\delta\mathcal{W}_{\text{iner}} = -\rho \iiint_V \delta\boldsymbol{q}^T_t \boldsymbol{M}(\boldsymbol{y}, \boldsymbol{z})\ddot{\boldsymbol{q}}_t\,\mathrm{d}V \tag{24}$$

$$= -\delta\boldsymbol{q}^T_{t_i} \underbrace{\left[\int \boldsymbol{N}^T\boldsymbol{M}\boldsymbol{N}\,\mathrm{d}x\right]}_{M^e} \ddot{\boldsymbol{q}}_{t_i} \tag{25}$$

where $\boldsymbol{M}^e$ is the element mass matrix

$$\boldsymbol{M}^e = \int_0^{\ell_e} \boldsymbol{N}^T\boldsymbol{M}\boldsymbol{N}\,\mathrm{d}x \tag{26}$$

ρ is the mass density, $\boldsymbol{N}$ is the shape function matrix, matrix $\boldsymbol{M}$ is

$$\boldsymbol{M} = \begin{bmatrix} I_0 & 0 & 0 & I_1 \\ 0 & I_0 & 0 & 0 \\ 0 & 0 & J_x & 0 \\ I_1 & 0 & 0 & I_2 \end{bmatrix} \tag{27}$$

and the coefficients of matrix (27) are defined as follows:

$$I_r = b_0 \sum_{k=1}^{N_{\text{lam}}} \rho^k \left\{\frac{z^{(r+1)}_{k+1} - z^{(r+1)}_k}{r+1}\right\}, \qquad r = 0, 1, 2$$

$$J_x = \frac{b^2_0}{12} I_0 + I_2$$

Note that the availability of symbolic manipulator such as MATHEMATICA® (Wolfram Research, Inc.) has made it possible to determine the analytical expressions for coefficients of both mass and stiffness matrices. When these matrices are obtained analytically, CPU time is reduced. This greatly saves CPU time for the Monte Carlo simulation because one no longer has to perform numerical integration for each case.

Free Vibration Analysis

Apply the dynamic version of the principle of virtual work,

$$\int_0^T \{\delta\mathcal{W}_{\text{ext}} - \delta\mathcal{W}_{\text{int}} - \delta\mathcal{W}_{iner}\}\,\mathrm{d}t = 0 \tag{28}$$

and further assume a harmonic response; the solution of these equations results in an eigenvalue problem:

$$\left[\boldsymbol{K}_g - \omega^2\boldsymbol{M}_g\right]\boldsymbol{q} = 0 \tag{29}$$

where $\boldsymbol{M}_g$ is the global mass matrix and $\boldsymbol{K}_g$ is the global stiffness matrix.

Probabilistic Approach

Overview

The present analysis assumes that the all random variables obey a normal distribution[18]:

$$f_i(x) = \left(1\big/\sigma_i\sqrt{2\pi}\right)\exp\left\{-\tfrac{1}{2}[(x-\mu_i)/\sigma_i]^2\right\} \tag{30}$$

where σ_i^2 and μ_i are the variance and the mean value of the ith random variable, respectively. In the present work, the random variables are considered as independent and are denoted as

$$\boldsymbol{x} = \{\theta_1, \theta_2, \ldots, \theta_n\} \tag{31}$$

where θ_i are the ply angles. Because these are independent random variables, the PDF can be expressed as follows:

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_i(x) \tag{32}$$

Several probabilistic methods have been used to analyze an uncertain unsymmetrically laminated beam by integrating uncertain aspects into the finite element modeling such as the perturbation technique using Taylor series expansion and simulation methods, that is, Monte Carlo simulation. Vinckenroy et al.[12] presented a new technique to analyze these structures by combining the stochastic analysis and the finite element method in structural design. However they did not extend their work to dynamic problems. Stochastic methods were also studied by Haldar and Mahadevan.[19] They applied the concepts to reliability analysis using the finite element method. The probabilistic methods used here are a subset of the stochastic methods. The only difference is that stochastic methods consider spatially correlated random variables and in probabilistic methods these random variables are not necessarily spatially correlated. Here we use the probabilistic finite element method with uncorrelated random variables.

Monte Carlo Simulation

Monte Carlo simulation (MCS), although computationally expensive, is a quite versatile technique that is capable of handling situations where other methods fail. The MCS is also often used to verify the results obtained from other methods.

MCS methods are based on the use of random variables and probability statistics to investigate problems. Most computers have the capability to generate uniformly distributed random numbers u_i between 0 and 1. Afterward random variables x_i are, in general, obtained as

$$s_i = s_i(x_i) = \Phi^{-1}(u_i) \tag{33}$$

where s_i is a function of the random variable x_i and Φ^{-1} is the inverse of the cumulative distribution function.

When working with normal distributed random variables, one faces the challenge of not having the inverse of the normal distribution function in a simple closed-form expression.[20] Therefore, random variables x_i were generated using a numerical method developed by Odeh and Evans.[21] However, other methods can be used such as the one proposed by Atkinson and Pearce.[22]

A large sample is generated, and then using PDFs, one evaluates the probability of having such values. The larger the number of simulations is, the higher the confidence is in the probability distribution of the obtained results. Therefore, for the present analysis, at least 10,000 realizations of the uncertain beam are performed, which increases the accuracy of the ply-angle distribution fitted to the sample data.

Probabilistic Eigenvalue Analysis

The probabilistic eigenvalue problem is expressed as

$$[\boldsymbol{K} - \lambda\boldsymbol{M}]\{\phi\} = 0 \tag{34}$$

where $\boldsymbol{K}$, $\boldsymbol{M}$, $\boldsymbol{\lambda}$, and ϕ are the probabilistic stiffness matrix, mass matrix, eigenvalues, and eigenvectors, respectively.

In problems where uncertainties are considered, there exists no density function describing the random nature of the system. The information is limited to only the mean values of the random variables. In such cases, perturbations techniques are suggested, among other existing techniques.[9,23] In general, the relationship between the random variables r_i and the matrix $\boldsymbol{Y}$ can be represented as a function of random variables:

$$\boldsymbol{Y} = \boldsymbol{Y}(x_1, x_2, \ldots, x_n) \tag{35}$$

In most cases the sensitivity derivatives of matrix $\boldsymbol{Y}$ can be obtained. Therefore, the matrix $\boldsymbol{Y}$ can be expanded using Taylor series expansion about the mean values (see Ref. 24):

$$\boldsymbol{Y} = \boldsymbol{Y}^0 + \sum_{i=1}^{n}\boldsymbol{Y}_i^I\epsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{Y}_{ij}^{II}\epsilon_i\epsilon_j + \cdots \tag{36}$$

$$\boldsymbol{Y}^0 = \boldsymbol{Y}\bigg|_{x=x^0} \quad \boldsymbol{Y}_i^I = \frac{\partial\boldsymbol{Y}}{\partial x_i}\bigg|_{x=x^0} \quad \boldsymbol{Y}_{ij}^{II} = \frac{\partial^2\boldsymbol{Y}}{\partial x_i\partial x_j}\bigg|_{x=x^0}$$

where $\boldsymbol{x}^0 = (\mu_1, \mu_2, \ldots, \mu_n)$ is a set of mean random variables and $\epsilon_i = x_i - \mu_i$ is a set of zero-mean uncorrelated random variables. For the case of random variables, ϵ_i is small, which allows us to approximate the series to second order and to ignore all higher-order terms.

In the free vibrational analysis of the present problem, the random nature of the stiffness matrix, mass matrix, eigenvalues, and eigenvectors are studied using a Taylor series expansion as mentioned earlier. Oh and Librescu[14] used a similar formulation for a stochastic Rayleigh–Ritz approach. Opposed to the work done by Oh and Librescu, the present formulation can use results provided by commercial finite element codes, greatly reducing the number of calculations. The present formulation also calculates the eigenvalue sensitivities up to second-order approximation.

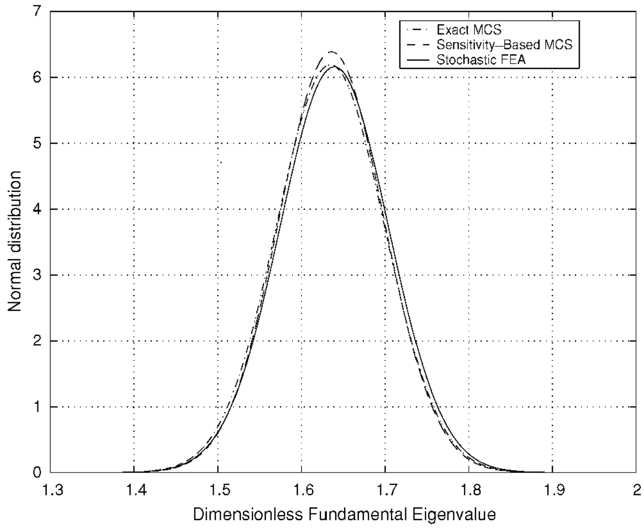
The presence of structural uncertainties results in certain randomness in the extensional matrix $\boldsymbol{A}$, extensional–bending coupling matrix $\boldsymbol{B}$, and bending matrix $\boldsymbol{D}$. The coefficients of these matrices are present in the equivalent bending–stiffness matrix $\boldsymbol{D}_c$ [Eq. (21)]. Thus, the matrix $\boldsymbol{D}_c$ will have certain randomness associated as well, and, by virtue of Eq. 23, these uncertainties affect the stiffness matrix. Thus, the random nature of the stiffness matrix $\boldsymbol{K}$ is expanded in terms of the mean-centered zeroth-, first-, and second-order rates of change with respect to the random variables as in Eq. (36):

$$\boldsymbol{K} = \boldsymbol{K}^0 + \sum_{i=1}^{n}\boldsymbol{K}_i^I\varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{K}_{ij}^{II}\varepsilon_i\varepsilon_j \tag{37}$$

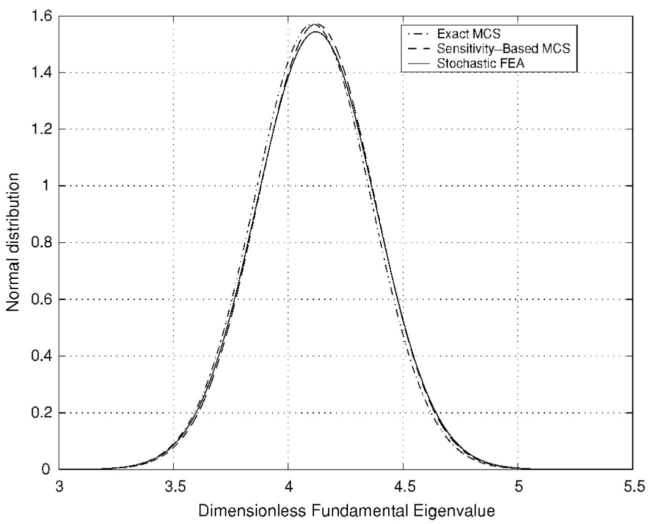
Similarly, the random nature of the mass matrix $\boldsymbol{M}$ is expanded in terms of the mean-centered zeroth-, first-, and second-order rates of change with respect to the random variables as

$$\boldsymbol{M} = \boldsymbol{M}^0 + \sum_{i=1}^{n}\boldsymbol{M}_i^I\varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{M}_{ij}^{II}\varepsilon_i\varepsilon_j \tag{38}$$

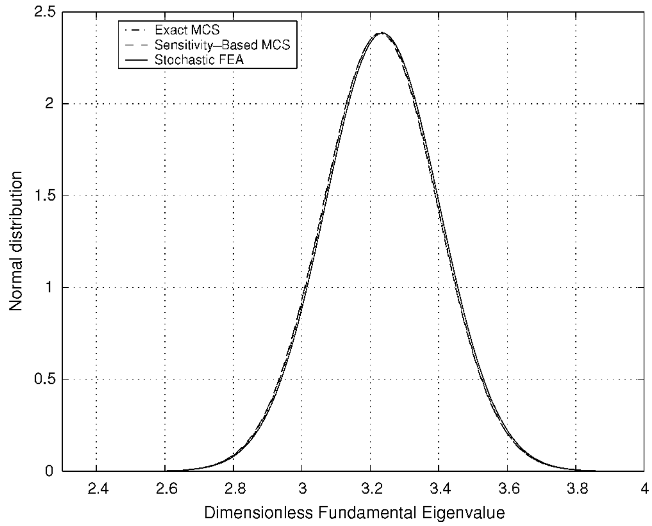
The eigenvalues and eigenvectors are also affected by uncertainties. The perturbed eigenvalues and eigenvectors are expressed in



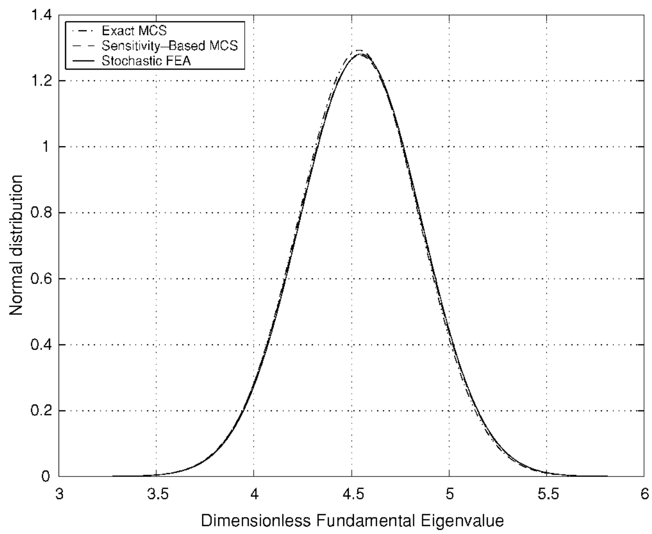
a) Four-ply symmetrical layup, 90/− 45/− 45/90 laminated beam



b) Four-ply unsymmetrical layup, 90/− 45/0/25 laminated beam



c) Eight-ply symmetrical layup, 90/− 45/30/0/0/30/− 45/90 laminated beam



d) Eight-ply unsymmetrical layup, 90/− 45/30/0/25/45/− 90/− 30 laminated beam

Fig. 1 Fundamental dimensionless eigenvalue distribution using exact MCS, sensitivity MCS, and SFEA for a cantilevered laminated beams.

terms of their mean-centered zeroth-, first-, and second-order rates of change with respect to the random variables as

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}^0 + \sum_{i=1}^{n} \boldsymbol{\lambda}_i^I \varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\lambda}_{ij}^{II} \varepsilon_i \varepsilon_j \tag{39}$$

The eigenvectors are

$$\{\phi\} = \{\phi^0\} + \sum_{i=1}^{n} \{\phi_i^I\}\varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \{\phi_{ij}^{II}\}\varepsilon_i \varepsilon_j \tag{40}$$

After Eqs. (37–40) are substituted into Eq. (34), the stochastic eigenvalue problem is expressed as

$$\begin{aligned}
&\left[\boldsymbol{K}^0 + \sum_{i=1}^{n} \boldsymbol{K}_i^I \varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{K}_{ij}^{II} \varepsilon_i \varepsilon_j\right] \\
&\quad\times \left\{\{\phi^0\} + \sum_{i=1}^{n} \{\phi_i^I\}\varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \{\phi_{ij}^{II}\}\varepsilon_i \varepsilon_j\right\} \\
&\quad= \left(\boldsymbol{\lambda}^0 + \sum_{i=1}^{n} \boldsymbol{\lambda}_i^I \varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\lambda}_{ij}^{II} \varepsilon_i \varepsilon_j\right) \\
&\quad\times \left[\boldsymbol{M}^0 + \sum_{i=1}^{n} \boldsymbol{M}_i^I \varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{M}_{ij}^{II} \varepsilon_i \varepsilon_j\right] \\
&\quad\times \left\{\{\phi^0\} + \sum_{i=1}^{n} \{\phi_i^I\}\varepsilon_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \{\phi_{ij}^{II}\}\varepsilon_i \varepsilon_j\right\}
\end{aligned} \tag{41}$$

The uncertainties in the random variables are in general small. As a consequence, in the applied perturbation technique, it is sufficient to consider only up to second-order terms. The expansion of Eq. (41) leads to three equations, which are solved successively.

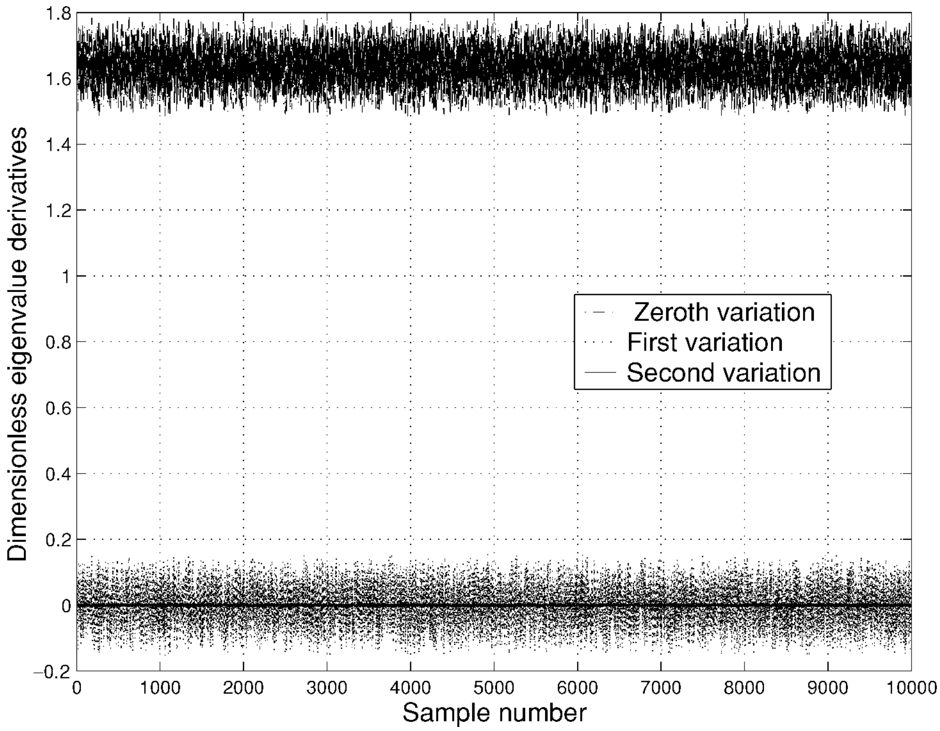
When the zeroth-order terms of ε_i is equated in the expansion, an eigenvalue problem for the mean-valued system is obtained. Therefore, the mean-centered zeroth derivative eigenvalues and associated eigenvectors are obtained from

$$[\boldsymbol{K}^0 - \boldsymbol{\lambda}^0 \boldsymbol{M}^0]\{\phi^0\} = 0 \tag{42}$$

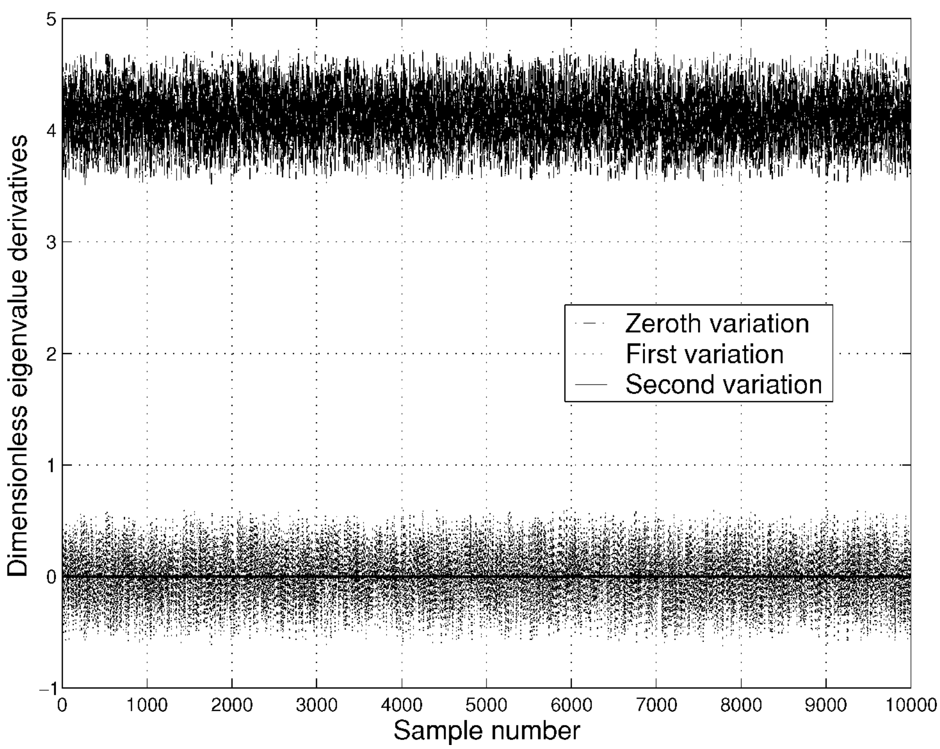
After premultiplying the first- and second-order terms of ε_i by $\{\phi^0\}^T$, the expressions for the mean-centered first and second eigenvalue derivatives are found as

$$\boldsymbol{\lambda}_i^I = \frac{\{\phi^0\}^T\left[\boldsymbol{K}_i^I - \boldsymbol{\lambda}^0 \boldsymbol{M}_i^I\right]\{\phi^0\}}{\{\phi^0\}^T[\boldsymbol{M}^0]\{\phi^0\}} \tag{43}$$

$$\boldsymbol{\lambda}_{ij}^{II} = \frac{\{\phi^0\}^T\left[\boldsymbol{K}_{ij}^{II} - \boldsymbol{\lambda}^0 \boldsymbol{M}_{ij}^{II} - 2\boldsymbol{\lambda}_i^I \boldsymbol{M}_j^I\right]\{\phi^0\}}{\{\phi^0\}^T[\boldsymbol{M}^0]\{\phi^0\}} \tag{44}$$



a) Symmetrical layup, 90/− 45/− 45/90 laminated beam



b) Unsymmetrical layup, 90/− 45/0/25 laminated beam

Fig. 2 Sensitivity of the fundamental dimensionless eigenvalue for a 4-ply cantilevered laminated beam to the first and second derivative contributions; note that the contribution of the second derivative can be neglected. Zeroth variation is represented the darkest lines.

Moreover, the mass matrix is not affected when only considering those uncertainties involving ply orientations. Therefore, the derivatives of the mass matrix $\boldsymbol{M}$ with respect to ply angles vanish, which reduces Eq. (38) to the deterministic mass matrix $\boldsymbol{M}^0$. This simplifies the preceding expressions for the mean-centered first and second eigenvalue derivatives.

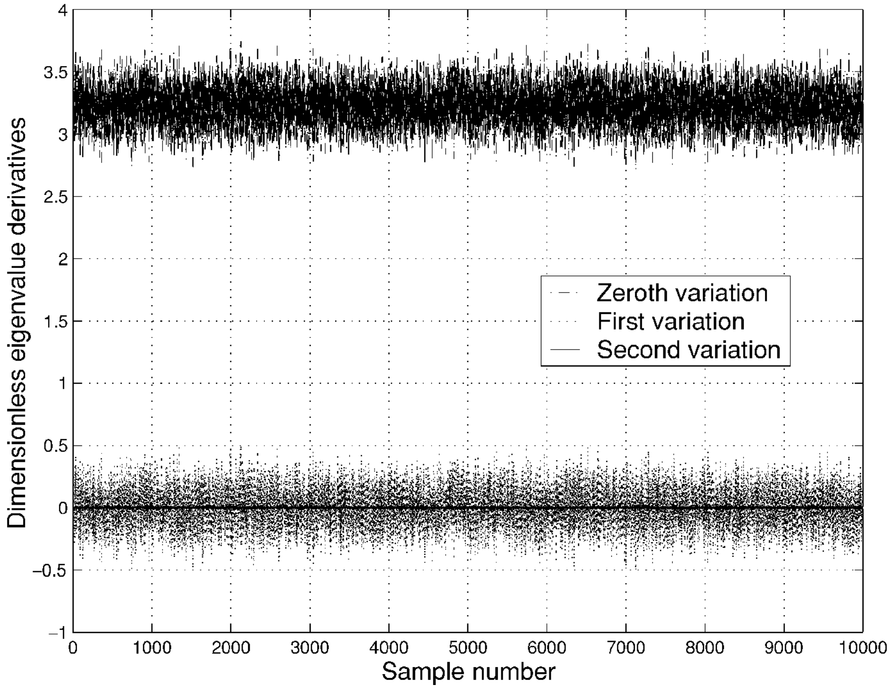
The advantage of the perturbation method is that the eigenvalue problem needs to be solved only once. The sensitivity analysis is done by using results from the mean-valued eigenvalue problem. This results in great computational saving.

The mean value of the eigenvalue is obtained by taking the expected value of Eq. (39):

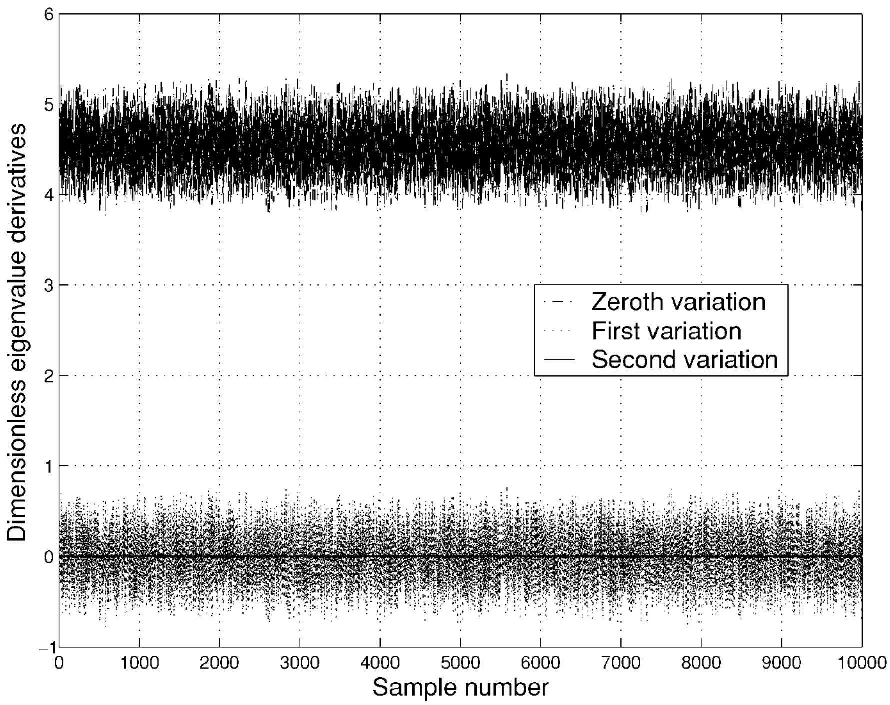
$$\mu_\lambda = E[\lambda] = \boldsymbol{\lambda}^0 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\lambda}_{ij}^{II} E[\varepsilon_i \varepsilon_j] \tag{45}$$

The variance and the standard deviation are defined as

$$\mathrm{var}[\lambda] = E[\lambda^2] - \mu_\lambda^2 \tag{46}$$



a) Symmetrical layup, 90/− 45/30/0/0/30/− 45/90 laminated beam



b) Unsymmetrical layup, 90/− 45/30/0/25/45/− 90/− 30 laminated beam

Fig. 3 Sensitivity of the fundamental dimensionless eigenvalue for a 8-ply cantilevered laminated beam to the first and second derivative contributions; note that the contribution of the second derivative can be neglected. Zeroth variation is represented by the darkest lines.

$$\sigma_\lambda = \sqrt{\mathrm{var}[\lambda]} \tag{47}$$

The uncertainty associated with the inherent randomness is given by the coefficient of variation[23]

$$\delta_\lambda = \sigma_\lambda / \mu_\lambda \tag{48}$$

For symmetrically distributed independent random variables,

$$\mathrm{var}[\lambda] = \sum_{i=1}^{n} \boldsymbol{\lambda}_i^I \boldsymbol{\lambda}_i^I E\left[\varepsilon_i^2\right] + \frac{1}{4} \sum_{i=1}^{n} \sum_{k=1}^{n} \boldsymbol{\lambda}_{ii}^{II} \boldsymbol{\lambda}_{kk}^{II} \left\{ E\left[\varepsilon_i^2 \varepsilon_k^2\right] - E\left[\varepsilon_i^2\right] E\left[\varepsilon_k^2\right] \right\} \tag{49}$$

where

$$E\left[\varepsilon_i^2\right] = \sum_{q=1}^{N_{\mathrm{sam}}} \frac{(x_q - \mu_i)^2}{N_{\mathrm{sam}}}$$

$$E\left[\varepsilon_i^2 \varepsilon_k^2\right] = \sum_{q=1}^{N_{\mathrm{sam}}} \frac{(x_q - \mu_i)^2 (x_q - \mu_k)^2}{N_{\mathrm{sam}}}$$

$$E\left[\varepsilon_i^2\right] E\left[\varepsilon_k^2\right] = \sum_{q=1}^{N_{\mathrm{sam}}} \sum_{r=1}^{N_{\mathrm{sam}}} \left[\frac{(x_q - \mu_i)(x_r - \mu_k)}{N_{\mathrm{sam}}} \right]^2$$

and N_{sam} is the number of samples, equal to 10,000 in this work.

Calculating Derivatives

Hasselman and Hart[25] proposed a method to calculate derivatives of eigenvalues for reduced systems. However, they assumed that the contribution of the partial derivatives of the transformation matrix to the partial derivatives of the eigenvalues and eigenvectors is small.

When the derivatives of the eigenvalues are calculated, only derivatives of the stiffness and mass matrices are needed. The derivatives of the stiffness matrix are more involved because they require taking the derivatives of the equivalent bending–stiffness matrix [Eq. (21)]. Various numerical schemes, such as finite difference, among others, exist to evaluate these derivatives. When some of these numerical schemes are used, ill conditioning could be a problem. This problem can be avoided by the following formulation, which allows the derivatives to be obtained exactly by numerical multiplication. The technique consists of taking the derivatives of Eq. (20) with respect to each of the variables x_i:

$$[D_R]_{,x_i} = [D_{I,I}]_{,x_i} - [D_{I,II}]_{,x_i}[D_{II,II}]^{-1}[D_{II,I}] - [D_{I,II}][D_{II,II}]^{-1}_{,x_i}[D_{II,I}] - [D_{I,II}][D_{II,II}]^{-1}[D_{II,I}]_{,x_i} \tag{50}$$

$$\begin{aligned}[D_R]_{,x_ix_j} = {} & [D_{I,I}]_{,x_ix_j} - [D_{I,II}]_{,x_ix_j}[D_{II,II}]^{-1}[D_{II,I}] \\ & - [D_{I,II}]_{,x_i}[D_{II,II}]^{-1}_{,x_j}[D_{II,I}] - [D_{I,II}]_{,x_i}[D_{II,II}]^{-1}[D_{II,I}]_{,x_j} \\ & - [D_{I,II}]_{,x_j}[D_{II,II}]^{-1}_{,x_i}[D_{II,I}] - [D_{I,II}][D_{II,II}]^{-1}_{,x_ix_j}[D_{II,I}] \\ & - [D_{I,II}][D_{II,II}]^{-1}_{,x_i}[D_{II,I}]_{,x_j} - [D_{I,II}]_{,x_j}[D_{II,II}]^{-1}[D_{II,I}]_{,x_i} \\ & - [D_{I,II}][D_{II,II}]^{-1}_{,x_j}[D_{II,I}]_{,x_i} - [D_{I,II}][D_{II,II}]^{-1}[D_{II,I}]_{,x_ix_j}\end{aligned} \tag{51}$$

The derivatives of $[D_{II,II}]^{-1}$ are calculated using the following matrix definition:

$$[D_{II,II}]^{-1}[D_{II,II}] = [I] \tag{52}$$

These derivatives are

$$[D_{II,II}]^{-1}_{,x_i} = -[D_{II,II}]^{-1}[D_{II,II}]_{,x_i}[D_{II,II}]^{-1} \tag{53}$$

$$\begin{aligned}[D_{II,II}]^{-1}_{,x_ix_j} = {} & -[D_{II,II}]^{-1}_{,x_i}[D_{II,II}]_{,x_j}[D_{II,II}]^{-1} \\ & - [D_{II,II}]^{-1}_{,x_j}[D_{II,II}]_{,x_i}[D_{II,II}]^{-1} \\ & - [D_{II,II}]^{-1}[D_{II,II}]_{,x_ix_j}[D_{II,II}]^{-1}\end{aligned} \tag{54}$$

Results and Discussion

The numerical results are obtained for a cantilevered laminated beam. The beam's mechanical and geometrical properties were assumed uniform throughout the beam. The material considered for the analysis was graphite–epoxy. These properties are given in Tables 1 and 2 for which results were produced.

Three models were developed: exact MCS, sensitivity-based MCS, and PFEA. For all three models, 10 beam elements were used and 4 different laminated beams are considered: 1) symmetrically laminated cantilever beam with 4 plies, 2) and unsymmetrically laminated cantilever beam with 4 plies, 3) symmetrically laminated cantilever beam with 8 plies, and 4) and unsymmetrically laminated cantilever beam with 8 plies.

When the effect of the uncertainties of random variables on the fundamental natural frequencies is studied, it is convenient to study their squared value, that is, eigenvalues. The statistical analysis for eigenvalues can be expressed as the variation of the first standard deviation about the mean value as follows:

$$\lambda = \mu_\lambda \pm \sigma_\lambda \tag{55}$$

$$\lambda = \mu_\lambda (1 \pm \delta_\lambda) \tag{56}$$

Table 1 Mean value and coefficient of variation for the first three dimensionless eigenvalues[a]

	μ_λ			δ_λ, %		
Mode	Exact MCS	Sensitivity MCS	SFEA	Exact MCS	Sensitivity MCS	SFEA
			[90/−45]$_s$ Ply orientation			
1	1.63468	1.63590	1.63925	3.94174	3.81729	3.95012
2	63.4141	63.4623	63.5888	3.83659	3.71583	3.84535
3	484.611	484.992	485.924	3.69899	3.58305	3.70819
			[90/−45/0/25] Ply orientation			
1	4.10664	4.12331	4.12107	6.18145	6.15573	6.26900
2	154.076	154.697	154.617	5.87879	5.85355	5.96109
3	1125.94	1130.40	1129.87	5.47214	5.44725	5.54740
			[90/−45/30/0]$_s$ Ply orientation			
1	3.23110	3.22896	3.23559	5.17099	5.18588	5.17371
2	122.606	122.540	122.782	4.96294	4.97753	4.96644
3	909.740	909.388	911.079	4.68154	4.69592	4.57623
			[90/−45/30/0/25/45/−90/−30] Ply orientation			
1	4.53781	4.54280	4.54736	6.79755	6.87307	6.84617
2	169.500	169.712	169.873	6.44018	6.51050	6.48532
3	1231.24	1233.02	1234.09	5.96122	6.02419	6.00139

[a] $E_{xx}/E_{yy} = 13.75$, $G_{xy}/E_{yy} = 0.55$, $G_{yz}/E_{yy} = 0.25$, $G_{xz}/E_{yy} = 0.25$, $h_0/b_0 = 0.5$, $L_0/h_0 = 30.0$, $E_{yy} = 9.42512 \times 10^9$ Pa, $\nu_{xy} = 0.30$, $b_0 = 0.1$ m, and $\rho = 1550.07$ kg/m^3.

Table 2 Mean value and coefficient of variation for the dimensionless fundamental natural frequency[a]

	μ_ω			δ_ω, %		
Ply orientation	Exact MCS	Sensitivity MCS	SFEA	Exact MCS	Sensitivity MCS	SFEA
[90/−45]$_s$	1.27855	1.27902	1.28033	1.97087	1.90865	1.97506
[90/−45/0/25]	2.02648	2.03059	2.03004	3.09073	3.07786	3.13450
[90/−45/30/0]$_s$	1.79753	1.79693	1.79877	2.58550	2.59294	2.58686
[90/−45/30/0/ /25/45/−90/−30]	2.13021	2.13138	2.13245	3.39876	3.43654	3.42308

[a] $E_{xx}/E_{yy} = 13.75$, $G_{xy}/E_{yy} = 0.55$, $G_{yz}/E_{yy} = 0.25$, $G_{xz}/E_{yy} = 0.25$, $h_0/b_0 = 0.5$, $L_0/h_0 = 30.0$, $E_{yy} = 9.42512 \times 10^9$ Pa, $\nu_{xy} = 0.30$, $b_0 = 0.1$ m, and $\rho = 1550.07$ kg/m^3.

The statistical analysis for natural frequencies is obtained by taking the square root on both sides of Eq. (56). When only the first-order approximation is considered, the following expression is obtained:

$$\omega_n = \mu_\omega(1 \pm \delta_\lambda/2) \tag{57}$$

$$\omega_n = \mu_\omega(1 \pm \delta_\omega) \tag{58}$$

The natural frequencies are nondimensionalized with respect to the deterministic quantities as follows:

$$\hat{\omega}_n = \left(\omega_n L_0^2 \big/ h_0\right)\sqrt{\rho/E_{yy}} \tag{59}$$

In general, the ply-angle uncertainties are ± 2.5 deg. However, results were produced for twice the ply orientation uncertainties, that is, from -5 to 5 deg. The statistical analysis is shown in Tables 1 and 2.

In all three models, the mean values and the coefficient of variations were close. However, the PFEA is conservative in the sense that it overestimates the variation of the natural frequencies. Exact MCSs would have been the most accurate approach but also a very expensive one. Therefore, the PFEA can be safely used.

The sensitivity-based MCS is an alternative approach that produces fairly good results and saves time. This approach produces very good results for only 1000 samples as opposed to 10,000 samples employed in the exact MCS.

Table 1 shows that uncertainties do not have a great influence on higher dimensionless eigenvalues. However, in most of engineering applications, the fundamental mode is the most important one. The fundamental dimensionless natural frequencies are tabulated in Table 2, and it can be seen that they are well predicted by the two proposed models and that the results are in good agreement with the exact MCS.

Figure 1 shows that the present models have a good correlation when compared to the exact MCS. Also, as the number of plies is increased, the models correlated even better with exact MCS.

Figures 2 and 3 shows that the eigenvalues are mainly sensitive to the zeroth- and first-order terms. The second-order terms do not influence the behavior of the eigenvalues. Therefore, the analysis would have been accurate if only the first-order approximation was considered. Figures 2 and 3 show that the variation of the first derivatives contributes significantly to the amount of uncertainty involved. Therefore, the ply angle uncertainties can play an important role in affecting free vibrations of symmetrically and unsymmetrically laminated beams. However, this influence for the fundamental mode is very small.

Conclusions

MCS has been applied to symmetric and unsymmetrically laminated beams with randomness in ply orientation to study the free vibrations. At least 10,000 realizations of the Monte Carlo sampling have been performed to improve the accuracy of the analysis.

A second sensitivity-based MCS has been developed using perturbation methods. With the use of Taylor series expansion, the eigenvalues has been expressed as mean-valued and probabilistic quantities. The accuracy of the results have been compared to those obtained by exact MCS.

A third approach, the probabilistic finite element approach was also developed. It proved to be conservative and gave a very good prediction of the behavior of the fundamental natural frequency in the presence of ply angle uncertainties.

The proposed two methods are advantageous over other techniques because the eigenvalue problem needs to be solved only once. Also, an elegant way to obtain sensitivity derivatives has been detailed.

Based on the results, the methods yield great computational savings when one is interested in predicting the statistics of the fundamental natural frequency of unsymmetrically laminated beam in the presence of ply-angle uncertainties.

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